

Central limit theorems for hyperbolic spaces and Jacobi processes on $[0, \infty[$

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Abstract

We present a unified approach to a couple of central limit theorems for radial random walks on hyperbolic spaces and time-homogeneous Markov chains on $[0, \infty[$ whose transition probabilities are defined in terms of the Jacobi convolutions. The proofs of all results are based on limit results for the associated Jacobi functions. In particular, we consider $\alpha \rightarrow \infty$, the case $\varphi_{i\rho-\lambda}^{(\alpha,\beta)}(t)$ for small λ , and $\varphi_{i\rho-n\lambda}^{(\alpha,\beta)}(t/n)$ for $n \rightarrow \infty$. The proofs of all these limit results are based on the known Laplace integral representation for Jacobi functions. Parts of the results are known, other improve known ones, and other are new.

KEYWORDS: Laplace integral representation, limits of Jacobi functions, asymptotic results, spherical functions, hyperbolic spaces, radial random walks, central limit theorems, normal limits, Rayleigh distributions.

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1 Introduction

We here derive a couple of central limit theorems for the distance of radial random walks $(S_n^k)_{n \geq 0}$ from their starting points on the hyperbolic spaces $H_k(\mathbb{F})$ of dimension $k \geq 2$ over the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or the quaternions \mathbb{H} . The main observation for proofs is that the distance processes are again Markov chains on $[0, \infty[$ whose transition probabilities are related to the product formula for the spherical functions for $H_k(\mathbb{F})$, i.e., certain Jacobi functions. As all proofs work without additional effort for more general “Jacobi random walks” on $[0, \infty[$, i.e., Markov processes on $[0, \infty[$ whose transition probabilities are related to general Jacobi functions, we shall derive all results in this context.

To describe the main results, we regard $H_k(\mathbb{F})$ as symmetric space G/K with

$$\begin{aligned} \mathbb{F} = \mathbb{R} : & \quad G = SO_o(1, k), \quad K = SO(k) \\ \mathbb{F} = \mathbb{C} : & \quad G = SU(1, k), \quad K = S(U(1) \times U(k)) \\ \mathbb{F} = \mathbb{H} : & \quad G = Sp(1, k), \quad K = Sp(1) \times Sp(k) \end{aligned}$$

and define the dimension parameter $d := \dim_{\mathbb{R}} \mathbb{F} = 1, 2, 4$. We identify the double coset space

$G//K$ with the interval $[0, \infty[$ such that $t \in [0, \infty[$ corresponds with the double coset

$$Ka_tK \quad \text{with} \quad a_t = \begin{pmatrix} \text{ch } t & 0 & \dots & 0 & \text{sh } t \\ 0 & & & & 0 \\ \vdots & & I_{k-1} & & \vdots \\ 0 & & & & 0 \\ \text{sh } t & 0 & \dots & 0 & \text{ch } t \end{pmatrix};$$

see e.g. [F] or [Hel]. Using this homeomorphism $\varphi : G//K \rightarrow [0, \infty[$, we define the hyperbolic distance on G/K by $d(xK, yK) = \varphi(Ky^{-1}xK)$. In this way, G acts on $G/K = H_k(\mathbb{F})$ isometrically in a two-point homogeneous way, i.e., for $x_1, x_2, y_1, y_2 \in H_k(\mathbb{F})$ with $d(x_1, x_2) = d(y_1, y_2)$ there exists $g \in G$ with $g(x_1) = y_1$ and $g(x_2) = y_2$.

Now consider a time-homogeneous Markov chain $(S_n^k)_{n \geq 0}$ on $H_k(\mathbb{F})$ with transition kernel K_k starting at time 0 in $eK \in G/K = H_k(\mathbb{F})$. This Markov chain is called a radial random walk on $H_k(\mathbb{F})$, if K_k is G -invariant, i.e., if for all $g \in G$, $x \in H_k(\mathbb{F})$, and Borel sets $A \subset H_k(\mathbb{F})$, $K_k(g(x), g(A)) = K_k(g, A)$. It is well-known (see e.g. Lemma 4.4 of [RV1]) that then for the canonical projection $\pi : G/H = H_k(\mathbb{F}) \rightarrow G//H = [0, \infty[$, the image process $(\pi(S_n^k))_{n \geq 0}$ is a time-homogeneous Markov chain on $[0, \infty[$ with kernel

$$\tilde{K}_k(x, A) = (\nu * \delta_x)(A) \quad \text{for } x \in [0, \infty[\text{ and Borel sets } A \subset [0, \infty[,$$

where $\nu \in M^1([0, \infty[)$ is the distribution of $d(S_n^k, S_{n+1}^k)$ (which is independent of n), δ_x is a point measure, and $*$ denotes the double coset convolution on $G//H = [0, \infty[$. We point out that the Markov kernel K_k and thus the finite-dimensional distributions of $(S_n^k)_{n \geq 0}$ are determined uniquely by $\nu \in M^1([0, \infty[)$, and that the distributions of the distances $d(S_n^k, eK)$ from the starting point eK are the n -th convolution powers $\nu^{(n)}$ of ν w.r.t. $*$.

We are now interested in central limit theorems (CLTs) for $d(S_n^k, eK)$ for $n \rightarrow \infty$.

For the first type of results, we fix $\nu \in M^1([0, \infty[)$ and $H_k(\mathbb{F})$, and introduce for each $c \in]0, 1[$ the compressing map $D_c : x \mapsto cx$ on $[0, \infty[$ as well as the compressed measure $\nu_c := D_c(\nu) \in M^1([0, \infty[)$. Now consider the radial random walk $(S_n^{(k,c)})_{n \geq 0}$ on $H_k(\mathbb{F})$ associated with the compressed measure ν_c . We now look for CLTs for $(S_n^{(k,n^{-r})})_{n \geq 0}$ depending on $r \geq 0$ which include or supplement the results in [KTS], [Tu], [Gr], Section 3.2 of [Te], [Tr], and in [Z1], [Z2], and Section 7.4 of the monograph [BH] in our setting.

The most classical case appears for $r = 1/2$ and is well-known; see Theorem 7.4.1 of [BH] and references cited there. Here, the distance processes $\frac{1}{\sqrt{n}}d(S_n^{(k,n^{-r})}, eK)$ from the starting point tend in distribution to some known limit depending on $H_k(\mathbb{F})$ and the second moment of ν . These limit distributions are known as (radial parts) of Gaussians on $H_k(\mathbb{F})$. As this case is studied precisely in the literature, we omit this case here.

The case $r = 0$ without initial compression is due to [Z1]; see Section 7.4 of [BH]. We here improve this CLT by a rate of convergence. The CLT needs the modified moment functions

$$m_j(t) := m_j(d, k; t) := \int_0^1 \int_0^\pi (\ln(|\text{ch } t + r \cdot e^{i\varphi} \text{sh } t|))^j dm_{dk/2-1, d/2-1}(r, \varphi) \quad (j \in \mathbb{N})$$

with the probability measure $m_{dk/2-1, d/2-1} \in M^1([0, 1] \times [0, \pi])$ defined below in (1.4). It will turn out in (2.4) below that this definition agrees with that in [Z1] and [BH]. It is known that $m_1 \geq 0$ and $m_1(x)^2 \leq m_2(x) \leq x^2$ for $x \geq 0$ with equality precisely for $x = 0$.

1.1 Theorem. Let $\nu \in M^1([0, \infty[)$ with $\nu \neq \delta_0$ and finite second moment. For $j \in \mathbb{N}$ let $M_j := \int_0^\infty m_j(t) d\nu(t)$ be the modified moments of ν with $M_1, M_2 < \infty$ by our assumption. Then

$$\frac{d(S_n^{(k,1)}, eK) - nM_1}{\sqrt{n}}$$

tends in distribution to $N(0, M_2 - M_1^2)$ with $M_2 - M_1^2 > 0$.

Moreover, if $\nu \in M^1([0, \infty[)$ in addition has a finite third moment, then the distribution functions of the random variables above tend uniformly on \mathbb{R} to the distribution function of $N(0, M_2 - M_1^2)$ of order $n^{-1/3}$.

The case $r > 1/2$ was studied in [V2]. We reprove the result here in a different way:

1.2 Theorem. Let $\nu \in M^1([0, \infty[)$ with $\nu \neq \delta_0$ and finite second moment $m_2 := \int_0^\infty x^2 d\mu(x)$ (which is automatically positive). Then

$$\left(\frac{dk}{m_2}\right)^{1/2} \cdot n^{r-1/2} \cdot d(S_n^{(k,n^{-r})}, eK)$$

tends in distribution to the Rayleigh distribution $\rho_{dk/2-1}$, where ρ_α has the Lebesgue density

$$\frac{1}{2^\alpha \Gamma(\alpha + 1)} x^{2\alpha+1} e^{-x^2/2} \quad (x \geq 0).$$

Notice that in the preceding result $\dim_{\mathbb{R}} H_k(\mathbb{F}) = dk$ and that $\rho_{dk/2-1}$ is precisely the radial part of a dk -dimensional standard normal distribution on \mathbb{R}^{dk} . The preceding theorem therefore means that for $r > 1/2$, the CLT forgets the curvature and admits the behavior of classical sums of i.i.d. random variables on the tangent space.

The case $r \in]0, 1/2[$ was not considered before. We here obtain:

1.3 Theorem. Let $\nu \in M^1([0, \infty[)$ with $\nu \neq \delta_0$, with compact support, and thus with finite moments $m_l := \int_0^\infty x^l d\mu(x) \in]0, \infty[$ ($l \in \mathbb{N}$). Then:

(1) For $r \in]1/6, 1/2[$,

$$\frac{d(S_n^{(k,n^{-r})}, eK) - \frac{(d(k+1)/2-1)m_2 n^{1-2r}}{dk}}{n^{1/2-r}}$$

tends in distribution to $N(0, \frac{m_2}{dk})$.

(2) If $r = 1/6$, then the random variables of part (1) tend in distribution to $N(-M, \frac{m_2}{dk})$ with the mean

$$M := -\frac{(d(k+1)/2-1)d(k+3)/2-2 \cdot m_4}{6dk(dk/2+1)}.$$

(3) If $r \in]0, 1/6[$, then

$$\frac{d(S_n^{(k,n^{-r})}, eK) - \frac{(d(k+1)/2-1)m_2}{dk} \cdot n^{1-2r}}{n^{1-4r}} \longrightarrow M$$

in probability with M as in (2).

Besides the preceding limit theorems for a fixed hyperbolic space, we also derive the following CLT for a fixed field \mathbb{F} , where the dimension k and the number n of steps tend to infinity. It generalizes a result in [V3]:

1.4 Theorem. Let $(k_n)_{n \geq 1} \subset \mathbb{N}$ be increasing with $\lim_{n \rightarrow \infty} n/k_n = 0$, and fix \mathbb{F} as above. Let $\nu \in M^1([0, \infty[)$ with finite second moment $\int_0^\infty x^2 d\nu(x)$, and consider the associated radial random walks $(S_n^k)_{n \geq 0}$ on $H_k(\mathbb{F})$ for $k \in \mathbb{N}$. Then, $m_j := \int_0^\infty (\ln(\operatorname{ch} x)) d\nu(x) < \infty$ exist for $j = 1, 2$, and

$$\frac{d(S_n^{k_n}, S_0^{k_n}) - nm_1}{\sqrt{n}}$$

tends in distribution for $n \rightarrow \infty$ to $N(0, m_2 - m_1^2)$.

An extension of this CLT without the restriction $\lim_{n \rightarrow \infty} n/k_n = 0$ was recently derived by Grundmann [G] by using completely different methods.

We now briefly describe the common roots of the proof of the preceding CLTs. We regard the spherical functions of the Gelfand pair (G, H) above as continuous functions on $[0, \infty[$ which are multiplicative w.r.t. $*$, i.e., $f(x)f(y) = \int_0^\infty f d(\delta_x * \delta_y)$ for $x, y \geq 0$. It is well-known (see [Ko2]) that in our case all spherical functions are given by Jacobi functions

$$\varphi_\lambda^{(\alpha, \beta)}(t) := {}_2F_1((\alpha + \beta + 1 - i\lambda)/2, (\alpha + \beta + 1 + i\lambda)/2; \alpha + 1; -\operatorname{sh}^2 t) \quad (\lambda \in \mathbb{C}) \quad (1.1)$$

with the parameters

$$\alpha = dk/2 - 1, \quad \beta = d/2 - 1 \quad \text{with} \quad d := \dim_{\mathbb{R}} \mathbb{F} = 1, 2, 4. \quad (1.2)$$

Moreover, the double coset convolutions $*$ on $[0, \infty[$ for the hyperbolic spaces above can be regarded as special cases of Jacobi convolution $*_{(\alpha, \beta)}$ on $[0, \infty[$ which were investigated mainly by Flensted-Jensen and Koornwinder. In the following we refer to the survey [Ko2] on the subject. For $\alpha > \beta \geq -1/2$ with $\alpha > -1/2$, this convolution is given by

$$\delta_s *_{(\alpha, \beta)} \delta_t(f) := \int_0^1 \int_0^\pi f(\operatorname{arch} |\operatorname{ch} s \cdot \operatorname{ch} t + r e^{i\varphi} \operatorname{sh} s \cdot \operatorname{sh} t|) dm_{\alpha, \beta}(r, \varphi) \quad (1.3)$$

for $f \in C_b([0, \infty[)$ and for the probability measure $dm_{\alpha, \beta}$ with

$$dm_{\alpha, \beta}(r, \varphi) = \frac{2\Gamma(\alpha + 1)}{\Gamma(1/2)\Gamma(\alpha - \beta)\Gamma(\beta + 1/2)} \cdot (1 - r^2)^{\alpha - \beta - 1} (r \sin \varphi)^{2\beta} \cdot r dr d\varphi \quad (1.4)$$

for $\alpha > \beta > -1/2$. For $\alpha > \beta = -1/2$, the measure degenerates into

$$dm_{\alpha, -1/2}(r, \varphi) = \frac{2\Gamma(\alpha + 1)}{\Gamma(1/2)\Gamma(\alpha + 1/2)} (1 - r^2)^{\alpha - 1/2} dr \cdot \frac{1}{2} d(\delta_0 + \delta_\pi)(\varphi), \quad (1.5)$$

and for $\alpha = \beta > -1/2$ into

$$dm_{\alpha, \alpha}(r, \varphi) = \frac{2\Gamma(\alpha + 1)}{\Gamma(1/2)\Gamma(\alpha + 1/2)} \sin^{2\alpha} \varphi d\varphi \cdot d\delta_0(r). \quad (1.6)$$

Now fix $\alpha \geq \beta \geq -1/2$ with $\alpha > -1/2$. It is well-known that the Jacobi convolution above can be extended uniquely in a weakly continuous, bilinear way to a probability-preserving convolution $*_{(\alpha, \beta)}$ on $M_b([0, \infty[)$, and that one obtains the so-called Jacobi-type hypergroups on $[0, \infty[$; see [Ko2], [BH], [Tr].

Using this Jacobi-convolution, we now generalize the Markov processes $(d(S_n^k, eK))_{n \geq 0}$ above as follows: Fix a measure $\nu \in M^1([0, \infty[)$, and consider a time-homogeneous Jacobi

random walk $(S_n^{(\alpha,\beta)})_{n \geq 0}$ on $[0, \infty[$ with law ν of index (α, β) , i.e., a time-homogeneous Markov process on $[0, \infty[$ starting at 0 with transition probability

$$P(S_{n+1}^{(\alpha,\beta)} \in A | S_n^{(\alpha,\beta)} = x) = (\delta_x *_{(\alpha,\beta)} \nu)(A) \quad (x \geq 0, A \subset [0, \infty[\text{ a Borel set}).$$

This notion agrees with that for $(d(S_n^k, eK))_{n \geq 0}$ in the hyperbolic case above. We shall derive all CLTs above in this more general setting. We shall do this in Section 3 for growing dimensions and in Section 4 for fixed dimension k . In this way, Theorems 1.2, 1.1, 1.3, and 1.4 are just special cases of Theorems 4.3, 4.2, 4.7 and 3.1 below respectively. The proofs of all these limit theorems will be based on several limit results for Jacobi functions which we will derive in Section 2. The basis of all these limit results will be the following well-known Laplace integral representation for the Jacobi functions; see Section 5.2 of [Ko2]:

1.5 Theorem. *Let $\alpha \geq \beta \geq -1/2$. Then, for $\lambda \in \mathbb{C}$ and $t \geq 0$,*

$$\varphi_\lambda^{(\alpha,\beta)}(t) = \int_0^1 \int_0^\pi |\text{ch } t + r e^{i\varphi} \text{sh } t|^{i\lambda - \rho} dm_{\alpha,\beta}(r, \varphi)$$

with

$$\rho := \alpha + \beta + 1 \geq 0$$

and the probability measure $dm_{\alpha,\beta}$ introduced in (1.4), (1.5) and (1.6) respectively.

We mention that Theorem 1.5 admits an analogue for Jacobi polynomials due to Koornwinder [Ko1], and that this integral representation also leads to limit results. In particular, one can revisit Hilb's formula for Jacobi polynomials (see [Sz]) in order to prove CLTs for Markov chains on \mathbb{Z}_+ associated with orthogonal polynomials; see [Ga], [V1], Section 7.4 of [BH], and references cited there for the topic.

We also mention that parts of this paper can be extended to certain families of Heckman-Opdam hypergeometric functions of type BC which include the spherical functions for the symmetric spaces $SU(p, q)/(SU(p) \times SU(q))$. For the background on these functions and the associated convolution structures on Weyl chambers of type B we refer to [H], [HS], [O] and [R]. In [RV2] we generalize the Harish Chandra integral representation for $SO_0(p, q)/(SO(p) \times SO(q))$ for $\mathbb{F} = \mathbb{R}$ of [Sa] to the more general setting considered in [R]; this integral representation is similar to that in Theorem 1.5 and leads with more technical effort to multidimensional extensions of some of the result in the present paper.

2 Limit relations for Jacobi functions

We start with two major results where α or both parameters α, β converge to infinity.

2.1 Proposition. *Let $\beta \geq -1/2$. Then there exists a constant $C = C(\beta)$ such that for all $t \geq 0$, $\alpha > \max(\beta, 0)$, and $\lambda \in \mathbb{R}$*

$$\left| \varphi_{i\rho - \lambda}^{(\alpha,\beta)}(t) - e^{i\lambda \cdot \ln(\text{ch } t)} \right| \leq C \frac{|\lambda| \cdot \min(1, t)}{\alpha^{1/2}}.$$

Proof. Using Theorem 1.5, we consider the difference

$$R := \left| \varphi_{i\rho - \lambda}^{(\alpha,\beta)}(t) - e^{i\lambda \cdot \ln(\text{ch } t)} \right| = \left| \int_0^1 \int_0^\pi \left(|\text{ch } t + r e^{i\varphi} \text{sh } t|^{i\lambda} - |\text{ch } t|^{i\lambda} \right) dm_{\alpha,\beta}(r, \varphi) \right|,$$

which satisfies

$$R \leq \int_0^1 \int_0^\pi |g(re^{i\varphi}, t) - 1| dm_{\alpha, \beta}(r, \varphi)$$

with

$$g(re^{i\varphi}, t) := |1 + re^{i\varphi} \cdot \text{sh } t / \text{ch } t|^{i\lambda} = e^{i\lambda \cdot \ln(|1 + re^{i\varphi} \cdot \text{sh } t / \text{ch } t|)}.$$

As $|e^{ix} - 1| \leq \sqrt{2} \cdot |x|$ for $x \in \mathbb{R}$, we have

$$|g(re^{i\varphi}, t) - 1| \leq \sqrt{2} \cdot |\lambda| \cdot |\ln(|1 + re^{i\varphi} \cdot \text{sh } t / \text{ch } t|)|.$$

Moreover, as for $z \in \mathbb{C}$ with $|z| < 1$

$$|\ln(|1 + z|)| \leq |\ln(1 + z)| \leq |z| + |z|^2 + |z|^3 \dots = |z|/(1 - |z|),$$

and as $0 \leq \text{sh } t / \text{ch } t \leq \min(1, t)$ for $t \geq 0$, we obtain for $0 \leq r \leq 1$ and $t \geq 0$

$$|g(re^{i\varphi}, t) - 1| \leq \sqrt{2} \cdot |\lambda| \cdot \frac{r \cdot \text{sh } t / \text{ch } t}{1 - r \cdot \text{sh } t / \text{ch } t} \leq \sqrt{2} \cdot |\lambda| \cdot \min(1, t) \frac{r}{1 - r}.$$

Therefore, as $r \in [0, 1]$,

$$|g(re^{i\varphi}, t) - 1| \leq 2\sqrt{2} \cdot |\lambda| \cdot \min(1, t) \frac{r}{1 - r^2}.$$

Now consider the probability measure $m_{\alpha, \beta}$ for $\alpha > \beta > -1/2$ introduced in Theorem 1.5. We conclude that

$$R \leq 2\sqrt{2} \cdot |\lambda| \cdot \frac{2 \cdot \min(1, t) \cdot \Gamma(\alpha + 1)}{\Gamma(1/2)\Gamma(\alpha - \beta)\Gamma(\beta + 1/2)} \int_0^\pi \sin^{2\beta} \varphi d\varphi \cdot \int_0^1 (1 - r^2)^{\alpha - \beta - 2} r^{2\beta + 2} dr.$$

Using standard formulas for the beta-integrals on the right-hand side and finally

$$\Gamma(\alpha + 1)/\Gamma(\alpha + 1/2) = O(\sqrt{\alpha}) \quad (\alpha \rightarrow \infty),$$

we obtain

$$R \leq 8 \cdot |\lambda| \cdot \min(1, t) \frac{\Gamma(\beta + 3/2)}{\Gamma(\beta + 1)} \cdot \frac{\Gamma(\alpha + 1)}{(\alpha - \beta - 1)\Gamma(\alpha + 1/2)} = |\lambda| \cdot \min(1, t) \cdot O(1/\sqrt{\alpha})$$

as claimed. The case $\beta = -1/2$ follows in the same way from the definition of $m_{\alpha, \beta}$. \square

We now consider a variant where α and β tend to infinity in a coupled way. We note that the convergence results 2.1 and 2.2 (without error estimates) correspond to well-known limit transitions for Jacobi polynomials which can be found e.g. in [Ko3] or [Sz].

2.2 Proposition. *Fix constants $c > 1$ and $d > 0$ and put $\alpha := c\beta + d$. Then there exists a constant $C = C(c, d)$ such that for all $t \geq 0$, $\beta > 0$, and $\lambda \in \mathbb{R}$*

$$\left| \varphi_{i\rho - \lambda}^{(c\beta + d, \beta)}(t) - e^{i\lambda \cdot \ln \sqrt{\text{ch } 2t - (1/c)\text{sh } 2t}} \right| \leq C \frac{|\lambda| \cdot \min(1, t)}{\beta^{1/2}}.$$

The proof will be based on the following observation which is likely well-known.

2.3 Lemma. Consider a continuous function $f : [0, 1] \rightarrow [0, \infty[$ such that there exist $x_0 \in [0, 1]$ and constants $0 < c_1 \leq c_2 \leq 1$ such that $f(x) \leq 1 - c_2(x - x_0)^2$ holds for all $x \in [0, 1]$, and $f(x) \geq 1 - c_1(x - x_0)^2$ for all $x \in [0, 1]$ in a suitable neighborhood of x_0 . Moreover, let $g : [0, 1] \rightarrow [0, \infty[$ be continuous with $g(x_0) > 0$. Then, for continuous $n \rightarrow \infty$,

$$\int_0^1 |x - x_0| \cdot f^n(x) \cdot g(x) dx = O\left(\frac{1}{\sqrt{n}} \int_0^1 f^n(x) \cdot g(x) dx\right).$$

Proof. For n sufficiently large, we obtain by continuity arguments and omitting parts of the left hand integral that

$$\int_0^1 f^n(x) \cdot g(x) dx \geq \int_0^{1/\sqrt{c_1 n}} (1 - c_1 x^2)^n g(x_0)/2 dx = \frac{g(x_0)}{2\sqrt{c_1 n}} \int_0^1 (1 - y^2/n)^n dy$$

where the integral on the right hand side converges to some positive constant. On the other hand,

$$\begin{aligned} \int_0^1 |x - x_0| \cdot f^n(x) \cdot g(x) dx &= \left(\int_0^{x_0} + \int_{x_0}^1 \right) |x - x_0| \cdot f^n(x) \cdot g(x) dx \\ &\leq 2 \int_0^1 x(1 - c_2 x^2)^n \cdot h(x) dx \\ &= \frac{2}{n\sqrt{c_2}} \int_0^{\sqrt{c_2 n}} (1 - y^2/n)^n \cdot h(y/\sqrt{nc_2}) dy \end{aligned}$$

for some continuous function h depending on g where the integral on the right hand side converges to some finite positive constant. A combination of both results leads to the lemma. \square

Proof of Proposition 2.2. Precisely as in the proof of Proposition 2.1 we obtain that

$$R := \left| \varphi_{i\rho-\lambda}^{(\alpha,\beta)}(t) - e^{i\lambda \cdot \ln \sqrt{\text{ch } 2t - (1/c)\text{sh } 2t}} \right| \leq 2|\lambda| \int_0^1 \int_0^\pi |\ln A| dm_{\alpha,\beta}(r, \varphi)$$

for

$$A := \frac{|\text{ch } t + r e^{i\varphi} \text{sh } t|}{|\text{ch } t + i \cdot \frac{1}{\sqrt{c}} \text{sh } t|}. \quad (2.1)$$

As for $r \in [0, 1]$ and $c > 1$, we have $(1 - r)/2 \leq A \leq 1 + r$, we obtain

$$|A - 1| \leq \max(r, (r + 1)/2) = (r + 1)/2.$$

Therefore, using $|\ln(|1 + z|)| \leq |z|/(1 - |z|)$ for $|z| = |A - 1| \leq 1$ as in the preceding proof, we conclude that

$$|\ln A| \leq \frac{|A - 1|}{1 - |A - 1|} \leq \frac{2}{1 - r} \cdot \frac{|1 - A^2|}{|1 + A|} \leq \frac{2}{1 - r} \cdot |1 - A^2| \leq \frac{4}{1 - r^2} \cdot |1 - A^2|.$$

Moreover, defining $\tau := \text{sh } t / \text{ch } t \leq \min(t, 1)$, we have

$$\begin{aligned} |1 - A^2| &= \left| 1 - \frac{(1 + r\tau \cos \varphi)^2 + r^2 \tau^2 \sin^2 \varphi}{1 + \tau^2/c} \right| = \left| \frac{\tau^2(1/c - r^2) + 2r\tau \cos \varphi}{1 + \tau^2/c} \right| \\ &\leq \tau \left(2|\cos \varphi| + \left| \frac{1/c - r^2}{1 + \tau^2/c} \right| \right) \leq 2 \cdot \tau (|\cos \varphi| + |r - 1/\sqrt{c}|). \end{aligned}$$

In summary, we have

$$\begin{aligned}
R &\leq 16|\lambda|\tau \cdot \int_0^1 \int_0^\pi \frac{|\cos \varphi| + |r - 1/\sqrt{c}|}{1 - r^2} dm_{\alpha,\beta}(r, \varphi) \\
&= 16|\lambda|\tau \cdot \frac{2\Gamma(\alpha + 1)}{\Gamma(1/2)\Gamma(\alpha - \beta)\Gamma(\beta + 1/2)} \\
&\quad \cdot \int_0^1 \int_0^\pi (|\cos \varphi| + |r - 1/\sqrt{c}|) \cdot ((1 - r^2)r^{2/(c-1)})^{\beta(c-1)-2} \cdot r^{1+4/(c-1)}(1 - r^2)^d \cdot \sin^{2\beta} \varphi \, dr \, d\varphi.
\end{aligned}$$

We now apply Lemma 2.3 to $g(r) := r^{1+4/(c-1)}(1 - r^2)^d$ and $f(r) := (1 - r^2)r^{2/(c-1)}$ with the maximum value of f on $[0, 1]$ at $r_0 = 1/\sqrt{c}$ and notice that $dm_{\alpha,\beta}(r, \varphi)$ is a probability measure. This yields

$$\int_0^1 |r - 1/\sqrt{c}| \cdot ((1 - r^2)r^{2/(c-1)})^{\beta(c-1)-2} \cdot r^{1+4/(c-1)}(1 - r^2)^d \, dr = O(1/\sqrt{\beta}).$$

As a similar argument also yields

$$\int_0^\pi |\cos \varphi| \cdot \sin^{2\beta} \varphi \, d\varphi = O(1/\sqrt{\beta}),$$

we obtain $R \leq 16|\lambda|\tau \cdot O(1/\sqrt{\beta})$ as claimed. \square

We next turn to a limit concerning Bessel functions. Recapitulate that the normalized Bessel functions

$$j_\alpha(t) := {}_0F_1(\alpha + 1; -t^2/4) = \Gamma(\alpha + 1) \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (t/2)^{2n}}{n! \Gamma(n + \alpha + 1)}$$

with $j_\alpha(0) = 1$ for $\alpha > -1/2$ admit the integral representation

$$j_\alpha(t) = \frac{\Gamma(\alpha + 1)}{\Gamma(1/2)\Gamma(\alpha + 1/2)} \int_{-1}^1 e^{itu} (1 - u^2)^{\alpha-1/2} \, du. \quad (2.2)$$

In order to compare j_α with the Jacobi functions, we rewrite it as

$$j_\alpha(t) = \int_0^1 \int_0^\pi e^{itr \cos \varphi} dm_{\alpha,\beta}(r, \varphi) \quad (2.3)$$

for $\alpha > \beta \geq -1/2$. In fact, the right hand side of (2.3) can be easily reduced to (2.2) by applying first polar coordinates $u = r \cos \varphi$, $v = r \sin \varphi$ with $v \in [0, \sqrt{1 - u^2}]$ and then the transform $z := v/\sqrt{1 - u^2} \in [0, 1]$, where the z -integral fits the constants.

The following limit result is related to the asymptotic expansion of the Jacobi functions in terms of Bessel functions in [ST]; see Lemma 3.3 in [V2].

2.4 Proposition. *Let $\alpha \geq \beta \geq -1/2$ with $\alpha > -1/2$ and $T > 0$ a constant. There exists a constant $C = C(\alpha, \beta, T)$ such that for all $\lambda \in \mathbb{R}$, $t \in [0, T]$, and $n \geq 1$,*

$$|\varphi_{i\rho-n\lambda}^{(\alpha,\beta)}(t/n) - j_\alpha(\lambda t)| \leq C \cdot |\lambda|t^2/n.$$

Proof. The integral representations in Theorem 1.5 and (2.3) imply that

$$\begin{aligned} R &:= \left| \varphi_{i\rho-n\lambda}^{(\alpha,\beta)}(t/n) - j_\alpha(\lambda t) \right| \\ &\leq \int_0^1 \int_0^\pi \left| \exp(i\lambda n \cdot \ln |ch(t/n) + re^{i\varphi} \operatorname{sh}(t/n)|) - e^{i\lambda tr \cos \varphi} \right| dm_{\alpha,\beta}(r, \varphi). \end{aligned}$$

Using the well-known inequality $|e^{ix} - e^{iy}| \leq \sqrt{2} \cdot |x - y|$ for $x, y \in \mathbb{R}$, we obtain

$$R \leq |\lambda| \cdot \int_0^1 \int_0^\pi |n \cdot \ln |ch(t/n) + re^{i\varphi} \operatorname{sh}(t/n)| - tr \cos \varphi| dm_{\alpha,\beta}(r, \varphi).$$

As

$$\begin{aligned} \ln |ch(t/n) + re^{i\varphi} \operatorname{sh}(t/n)| &= \frac{1}{2} \ln ((ch(t/n) + r \cos \varphi \operatorname{sh}(t/n))^2 + r^2 \sin^2 \varphi \operatorname{sh}^2(t/n)) \\ &= \frac{1}{2} \ln (1 + 2r \cos \varphi \cdot t/n + t^2(1 + r^2)/n^2 + O(t^3/n^3)) \\ &= \frac{1}{2} \left(2r \cos \varphi \frac{t}{n} + \frac{t^2(1 + r^2)}{n^2} - 2r^2 \cos^2 \varphi \frac{t^2}{n^2} + O\left(\frac{t^3}{n^3}\right) \right) \end{aligned}$$

uniformly for $t \in [0, T]$, $r \in [0, 1]$, $\varphi \in [0, \pi]$, the claim follows. \square

The next result describes the oscillatory behavior of $\varphi_{i\rho-\lambda}^{(\alpha,\beta)}(t)$ in the spectral variable $\lambda \in \mathbb{R}$ for fixed α, β , which is uniform in $t \geq 0$. For this we follow Section 7.2.2 of [BH] and define for $k \in \mathbb{N}$ the so called moment functions

$$\begin{aligned} m_k(t) &:= m_k^{(\alpha,\beta)}(t) := \frac{\partial^k}{\partial \lambda^k} \varphi_{i\rho+i\lambda}^{(\alpha,\beta)}(t) \Big|_{\lambda=0} = \frac{\partial^k}{\partial \lambda^k} \varphi_{-i\rho-i\lambda}^{(\alpha,\beta)}(t) \Big|_{\lambda=0} \\ &= \int_0^1 \int_0^\pi (\ln(|ch t + r \cdot e^{i\varphi} \operatorname{sh} t|))^k dm_{\alpha,\beta}(r, \varphi) \end{aligned} \quad (2.4)$$

for $t \geq 0$ (where the second last equation follows from symmetry of the Jacobi functions in the parameter, and last one from Theorem 1.5). In particular, the function m_1 appears as substitute of the additive function $x \mapsto x$ on the group $(\mathbb{R}, +)$ on the Jacobi convolution structure $([0, \infty[, *_{\alpha,\beta})$; see [Z1] and Section 7.2 of the monograph [BH]. m_1 can be used to define a modified drift-part in a CLT on $[0, \infty[$; see [Z1] [BH] and Section 3 below. We mention that for the parameters α, β , for which the Jacobi functions are spherical functions of rank-one, non-compact symmetric spaces, this meaning of m_1 is well-known for a long time in probability theory on hyperbolic spaces; see, for instance, [KTS], [Tu], [F].

We now derive a result which improves a general result of Zeuner [Z1] for general Chebli-Trimeche hypergroups on $[0, \infty]$ in the special case of the Jacobi convolution structures $([0, \infty[, *_{\alpha,\beta})$. It will be used to derive a Berry-Esseen-type CLT below.

2.5 Proposition. *Let $\alpha \geq \beta \geq -1/2$ with $\alpha > -1/2$. Then there exists a constant $C = C(\alpha, \beta)$ such that for all $t \geq 0$ and $\lambda \in \mathbb{R}$,*

$$|\varphi_{i\rho-\lambda}^{(\alpha,\beta)}(t) - e^{i\lambda \cdot m_1(t)}| \leq C(\lambda^2 + |\lambda|^3).$$

The proof depends on the following elementary observation:

2.6 Lemma. For $z \in \mathbb{C}$ with $|z| \leq 1$, $\varepsilon \in]0, 1]$, and the Euler number $e = 2, 71\dots$,

$$|\ln |1 + z|| \leq \frac{1}{e\varepsilon(1 - |z|)^\varepsilon}.$$

Proof. Elementary calculus yields $|x^\varepsilon \cdot \ln x| \leq 1/(e\varepsilon)$ for $x \in]0, 1]$. Therefore,

$$\begin{aligned} |\ln |1 + z|| &= |\Re \ln(1 + z)| \leq |\ln(1 + z)| = |z - z^2/2 + z^3/3 \pm \dots| \\ &\leq |z| + |z|^2/2 + |z|^3/3 \pm \dots = |\ln(1 - |z|)| \leq \frac{1}{e\varepsilon(1 - |z|)^\varepsilon}. \end{aligned}$$

□

Proof of the Proposition: Let $h(t, r, \varphi) := |1 + re^{i\varphi} \cdot \text{sh } t / \text{ch } t|$. Then, for $t \geq 0$,

$$e^{i\lambda \cdot m_1(t)} = (\text{ch } t)^{i\lambda} \cdot \exp \left(i\lambda \int_0^1 \int_0^\pi \ln(h(t, r, \varphi)) \, dm_{\alpha, \beta}(r, \varphi) \right).$$

Therefore, using the integral representation of the Jacobi functions, we obtain

$$\begin{aligned} R &:= |\varphi_{i\rho - \lambda}^{(\alpha, \beta)}(t) - e^{i\lambda \cdot m_1(t)}| \\ &= \left| \int_0^1 \int_0^\pi e^{i\lambda \cdot \ln(h(t, r, \varphi))} \, dm_{\alpha, \beta}(r, \varphi) - \exp \left(i\lambda \int_0^1 \int_0^\pi \ln(h(t, r, \varphi)) \, dm_{\alpha, \beta}(r, \varphi) \right) \right|. \end{aligned}$$

We now write down the usual power series for both exponentials and observe that the terms of order 0 and 1 are equal in both expansions. Therefore,

$$\begin{aligned} R &\leq \int_0^1 \int_0^\pi \left| e^{i\lambda \cdot \ln(h(t, r, \varphi))} - (1 + i\lambda \cdot \ln(h(t, r, \varphi))) \right| \, dm_{\alpha, \beta}(r, \varphi) \\ &\quad + \left| \exp \left(i\lambda \int_0^1 \int_0^\pi \ln(h(t, r, \varphi)) \, dm_{\alpha, \beta}(r, \varphi) \right) - 1 - i\lambda \int_0^1 \int_0^\pi \ln(h(t, r, \varphi)) \, dm_{\alpha, \beta}(r, \varphi) \right|. \end{aligned}$$

Using the well-known estimates $|\cos x - 1| \leq x^2/2$ and $|\sin x - x| \leq |x|^3/6$ for $x \in \mathbb{R}$, we obtain $|e^{ix} - (1 + ix)| \leq x^2/2 + |x|^3/6$, and thus

$$R \leq A_1^2 \lambda^2 / 2 + A_1^3 |\lambda|^3 / 6 + A_2 \lambda^2 / 2 + A_3 |\lambda|^3 / 6$$

for $A_k := \int_0^1 \int_0^\pi |\ln(h(t, r, \varphi))|^k \, dm_{\alpha, \beta}(r, \varphi)$, $k = 1, 2, 3$. In particular, by Jensen's inequality,

$$R \leq A_2 \lambda^2 + A_3 |\lambda|^3 / 3. \quad (2.5)$$

Assume now that $\alpha > \beta > -1/2$ holds. Choose some $\varepsilon \in]0, 1[$ with $\varepsilon < (\alpha - \beta)/3$ and apply Lemma 2.6 as well as $1 - r \text{sh } t / \text{ch } t \geq 1 - r \geq (1 - r)/2$ for $r \in [0, 1], t \geq 0$. This and the definitions of A_k and h imply for $k = 2, 3$ and some constants C_1, C_2, C_3 that

$$\begin{aligned} A_k &\leq \int_0^1 \int_0^\pi \frac{C_1}{|1 - r \text{sh } t / \text{ch } t|^{k\varepsilon}} \, dm_{\alpha, \beta}(r, \varphi) \\ &\leq C_2 \int_0^1 \int_0^\pi (1 - r)^{\alpha - \beta - 1 - k\varepsilon} (r \sin \varphi)^{2\beta} \cdot r \, dr \, d\varphi \leq C_3 \end{aligned} \quad (2.6)$$

as claimed.

The case $\alpha > \beta = -1/2$ follows in the same way by using $m_{\alpha, -1/2}$.

Finally, the case $\alpha = \beta > -1/2$ can be reduced to the case $\alpha > \beta = -1/2$ by the well-known quadratic transform $\varphi_{2\lambda}^{(\alpha, \alpha)}(t) = \varphi_\lambda^{(\alpha, -1/2)}(2t)$ (see Eq. (5.32) of [Ko3]) which in particular implies $m_1^{(\alpha, \alpha)}(t) = \frac{1}{2} m_1^{(\alpha, -1/2)}(2t)$. □

2.7 Remark. The preceding proof leads easily to explicit constants $C = C(\alpha, \beta)$ in Proposition 2.5. For instance, for $\alpha > \beta + 1$, one may take $\varepsilon = 1/3$ above and obtains from (2.6) and explicit values of beta-integrals that $A_2, A_3 \leq \frac{6\alpha}{e(\alpha-\beta-1)}$ and thus, by (2.5), that

$$C := \frac{6\alpha}{e(\alpha - \beta - 1)}$$

is an admissible constant in the statement of Proposition 2.5.

We finally use the integral representation 2.4 of m_1 in order to estimate m_1 . Weaker estimates for m_1 for general Chebli-Trimeche hypergroups on $[0, \infty[$ are given in [Z2],[Z3]; see Proposition 7.3.23 of [BH]. A better estimation is derived in [G].

2.8 Lemma. *Let $\alpha \geq \beta \geq -1/2$ with $\alpha > -1/2$. Then there exists a constant $C = C(\alpha, \beta)$ with $t - C \leq m_1(t) \leq t$ for all $t \geq 0$.*

Proof. Let $t \geq 0$, $\varphi \in [0, \pi]$ and $r \in [0, 1]$. Then

$$e^t(1-r)/2 \leq |\operatorname{ch} t + r \cdot e^{i\varphi} \operatorname{sh} t| \leq \operatorname{ch} t + \operatorname{sh} t = e^t.$$

We conclude from (2.4) that

$$m_1(t) = \int_0^1 \int_0^\pi \ln(|\operatorname{ch} t + r \cdot e^{i\varphi} \operatorname{sh} t|) dm_{\alpha,\beta} \leq \int_0^1 \int_0^\pi \ln(e^t) dm_{\alpha,\beta} = t.$$

For the second inequality, we now assume $\alpha > \beta$ and conclude from Lemma 2.6 for $\varepsilon = (\alpha - \beta)/2$ that $\ln(1-r) \geq -C/(1-r)^\varepsilon$ and thus

$$m_1(t) \geq \int_0^1 \int_0^\pi \ln(e^t(1-r)/2) dm_{\alpha,\beta} \geq t - \ln 2 - C \cdot \int_0^1 \int_0^\pi (1-r)^{-\varepsilon} dm_{\alpha,\beta},$$

where the latter integral has a finite value. This implies the result for $\alpha > \beta$. The case $\alpha = \beta > -1/2$ can again be handled as in the end of the proof of Proposition 2.5. \square

This lemma and Proposition 2.5 lead to:

2.9 Corollary. *Let $\alpha \geq \beta \geq -1/2$ with $\alpha > -1/2$. Then there exists a constant $C = C(\alpha, \beta)$ such that for all $t \geq 0$ and $\lambda \in \mathbb{R}$,*

$$|\varphi_{i\rho-\lambda}^{(\alpha,\beta)}(t) - e^{i\lambda \cdot t}| \leq C(\lambda^2 + |\lambda|^3).$$

3 Central limit theorems for growing parameters

In this section we derive two CLTs for Jacobi random walks, where in the first result α tends to infinity with fixed β , while in the second one α and β tend to infinity.

3.1 Theorem. *Let $\beta \geq -1/2$ fixed, and let $(\alpha_n)_{n \geq 1} \subset [\beta, \infty[$ be an increasing sequence of parameters with $\lim_{n \rightarrow \infty} n/\alpha_n = 0$. Let $\nu \in M^1([0, \infty[)$ be a probability measure with a finite second moment $\int_0^\infty x^2 d\nu(x) < \infty$ and with $\nu \neq \delta_0$, and consider the associated Jacobi random walks $(S_n^{(\alpha_n, \beta)})_{n \geq 0}$ on $[0, \infty[$. Then*

$$\frac{S_n^{(\alpha_n, \beta)} - n \cdot m_1}{\sqrt{n}} \rightarrow N(0, m_2 - m_1^2)$$

in distribution for $n \rightarrow \infty$ with a normal distribution $N(0, m_2 - m_1^2)$ with parameters

$$m_1 := \int_0^\infty \ln(\operatorname{ch} x) d\nu(x) < \infty, \quad m_2 := \int_0^\infty (\ln(\operatorname{ch} x))^2 d\nu(x) \in]m_1^2, \infty[.$$

Proof. Consider the homeomorphism $T : [0, \infty[\rightarrow [0, \infty[$, $t \mapsto \ln \operatorname{ch} t$. We recapitulate from Proposition 2.2 that

$$\varphi_{i\rho-\lambda}^{(\alpha_n, \beta)}(t) = e^{i\lambda \cdot \ln \operatorname{ch} t} + O(|\lambda|/\sqrt{\alpha_n})$$

uniformly in $t \in [0, \infty[$. Therefore, there exists a constant $C > 0$ with

$$\left| \int_0^\infty \varphi_{i\rho-\lambda}^{(\alpha_n, \beta)}(t) d\nu^{(n; \alpha_n, \beta)}(t) - \int_0^\infty e^{i\lambda \cdot \ln \operatorname{ch} t} d\nu^{(n; \alpha_n, \beta)}(t) \right| \leq C \cdot \frac{|\lambda|}{\sqrt{\alpha_n}} \quad (3.1)$$

and

$$\left| \int_0^\infty \varphi_{i\rho-\lambda}^{(\alpha_n, \beta)}(t) d\nu(t) - \int_0^\infty e^{i\lambda \cdot \ln \operatorname{ch} t} d\nu(t) \right| \leq C \cdot \frac{|\lambda|}{\sqrt{\alpha_n}} \quad (3.2)$$

for $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$. Moreover, the random variables $T(S_n^{(\alpha_n, \beta)})$ have the distributions $T(\nu^{(n; \alpha_n, \beta)})$ with the classical Fourier transforms

$$T(\nu^{(n; \alpha_n, \beta)})^\wedge(\lambda) = \int_0^\infty e^{-i\lambda \cdot \ln \operatorname{ch} t} d\nu^{(n; \alpha_n, \beta)}(t).$$

Therefore, by (3.1) and (3.2),

$$\begin{aligned} T(\nu^{(n; \alpha_n, \beta)})^\wedge(\lambda) &= \int_0^\infty \varphi_{i\rho+\lambda}^{(\alpha_n, \beta)}(t) d\nu^{(n; \alpha_n, \beta)}(t) + O(|\lambda|/\sqrt{\alpha_n}) \\ &= \left(\int_0^\infty \varphi_{i\rho+\lambda}^{(\alpha_n, \beta)}(t) d\nu(t) \right)^n + O(|\lambda|/\sqrt{\alpha_n}) \\ &= \left(T(\nu)^\wedge(\lambda) + O(|\lambda|/\sqrt{\alpha_n}) \right)^n + O(|\lambda|/\sqrt{\alpha_n}). \end{aligned}$$

Moreover, as ν has a finite second moment by our assumption, and as $\ln \operatorname{ch} t \leq t$ for $t \geq 0$, the measure $T(\nu)$ also has finite first and second moments

$$m_k = \int_0^\infty t^k dT(\nu)(t) = \int_0^\infty (\ln \operatorname{ch} t)^k d\nu(t) \quad (k = 1, 2),$$

and thus

$$T(\nu)^\wedge(\lambda) = 1 - i\lambda m_1 - \lambda^2 m_2/2 + o(\lambda^2) \quad \text{for } \lambda \rightarrow 0.$$

Therefore, if we denote the distribution of $(T(S_n^{(\alpha_n, \beta)}) - n \cdot m_1)/\sqrt{n}$ by μ_n , and if we use the assumption $O(1/\sqrt{n\alpha_n}) = o(1/n)$, we conclude that for $\lambda \in \mathbb{R}$,

$$\begin{aligned} \mu_n^\wedge(\lambda) &= T(\nu^{(n; \alpha_n, \beta)})^\wedge(\lambda/\sqrt{n}) \cdot e^{in \cdot m_1 \lambda/\sqrt{n}} \\ &= \left(\left(T(\nu)^\wedge(\lambda/\sqrt{n}) + O(|\lambda|/\sqrt{n\alpha_n}) \right)^n + O(|\lambda|/\sqrt{n\alpha_n}) \right) \cdot e^{in \cdot m_1 \lambda/\sqrt{n}} \\ &= \left(1 - \frac{i\lambda m_1}{\sqrt{n}} - \frac{\lambda^2 m_2}{2n} + o(1/n) + O(|\lambda|/\sqrt{n\alpha_n}) \right)^n \cdot \left(1 + \frac{i\lambda m_1}{\sqrt{n}} - \frac{\lambda^2 m_1^2}{2n} + o(1/n) \right)^n \\ &= \left(1 - \frac{\lambda^2(m_2 - m_1^2)}{2n} + o(1/n) \right)^n, \end{aligned}$$

which tends for $n \rightarrow \infty$ to $e^{-\lambda^2(m_2 - m_1^2)/2} = N(0, m_2 - m_1^2)^\wedge(\lambda)$. The classical continuity theorem of Levy yields that $(T(S_n^{(\alpha_n, \beta)} - nm_1)/\sqrt{n})$ tends in distribution to $N(0, m_2 - m_1^2)$. This in particular shows that $\ln(\text{ch}(S_n^{(\alpha_n, \beta)}))/n \rightarrow m_1 > 0$ and thus $e^{-2S_n^{(\alpha_n, \beta)}} \rightarrow 0$ in probability. Using

$$x - \ln 2 \leq \ln(\text{ch } x) \leq x + \ln(1 + e^{-2x}) \leq x + e^{-2x}$$

and thus

$$\ln \text{ch } S_n^{(\alpha_n, \beta)} - e^{-2S_n^{(\alpha_n, \beta)}} \leq S_n^{(\alpha_n, \beta)} \leq \ln \text{ch } S_n^{(\alpha_n, \beta)} + \ln 2,$$

we obtain that $(S_n^{(\alpha_n, \beta)} - nm_1)/\sqrt{n}$ tends in distribution to $N(0, m_2 - m_1^2)$ as claimed. \square

3.2 Remark. The preceding theorem was derived in [V3] by completely different methods under the stronger condition $n/\sqrt{\alpha_n} \rightarrow 0$ for $n \rightarrow \infty$. Recently, the preceding theorem was generalized by W. Grundmann [G] to an arbitrary sequence $(\alpha_n)_n$ with $\alpha_n \rightarrow \infty$.

The following CLT can be proved in the same way as Theorem 3.1 by using Proposition 2.2 and the homeomorphism $T : [0, \infty[\rightarrow [0, \infty[$ with $T(x) := \ln \sqrt{\text{ch } 2x - (1/c)\text{sh } 2x}$ instead of Proposition 2.1 and $T(x) := \ln \text{ch } x$. We expect that it can also be generalized to an arbitrary sequence $(\alpha_n)_n$ with $\alpha_n \rightarrow \infty$ similar to [G].

3.3 Theorem. Fix constants $c > 1$ and $d > 0$, and let $(\beta_n)_{n \geq 1} \subset [\beta, \infty[$ be an increasing sequence of parameters with $\lim_{n \rightarrow \infty} n/\beta_n = 0$. Moreover, put $\alpha_n := c\beta_n + d$.

Let $\nu \in M^1([0, \infty[)$ be a probability measure with a finite second moment $\int_0^\infty x^2 d\nu(x) < \infty$ and with $\nu \neq \delta_0$, and consider the associated Jacobi random walks $(S_n^{(\alpha_n, \beta_n)})_{n \geq 0}$ on $[0, \infty[$. Then

$$\frac{S_n^{(\alpha_n, \beta_n)} - n \cdot m_1}{\sqrt{n}} \rightarrow N(0, m_2 - m_1^2)$$

in distribution for $n \rightarrow \infty$ with a normal distribution $N(0, m_2 - m_1^2)$ with parameters

$$m_1 := \int_0^\infty \ln \sqrt{\text{ch } 2x - (1/c)\text{sh } 2x} d\nu(x) > 0,$$

$$m_2 := \int_0^\infty (\ln \sqrt{\text{ch } 2x - (1/c)\text{sh } 2x})^2 d\nu(x) \in]m_1^2, \infty[.$$

4 Central limit theorems for fixed parameters

In this section we present a couple of CLTs for fixed parameters α, β . We consider the following setting: We fix some non-trivial probability measure $\nu \in M^1([0, \infty[)$ with $\nu \neq \delta_0$ which possibly satisfies some moment condition. For each $d \in]0, 1]$ consider the compressing map $D_d : x \mapsto dx$ on $[0, \infty[$ as well as the compressed measure $\nu_d := D_d(\nu) \in M^1([0, \infty[)$. For given ν and d we consider a Jacobi random walk $(S_n^{(\alpha, \beta, d)})_{n \geq 0}$ on $[0, \infty[$ associated with the law ν_d as above. We now investigate the limit behavior of $(S_n^{(\alpha, \beta, n^{-r})})_{n \geq 0}$ for different powers $r \geq 0$. The most classical cases appear for $r = 1/2$ and $r = 0$.

In fact, for $r = 1/2$, the random variables $S_n^{(\alpha, \beta, n^{-1/2})}$ tend in distribution to some probability measure $\gamma_{t_0}^{(\alpha, \beta)}$ which is part of the unique (up to time parametrization) Gaussian convolution semigroup $(\gamma_t^{(\alpha, \beta)})_{t \geq 0}$ on the Jacobi hypergroup on $[0, \infty[$ where the correct t_0

mainly depends on the second moment $m_2 := \int_0^\infty x^2 d\nu(x)$ of ν . For details see Theorem 7.4.1 of [BH]. We remark that this CLT holds for general Chebli-Trimeche hypergroups on $[0, \infty[$. As the proof is very standard and universal, we do not treat this case here.

The case $r = 0$ was handled by Zeuner [Z1] for Chebli-Trimeche hypergroups on $[0, \infty[$ with exponential growth by using an estimation weaker than Proposition 2.5. We here reprove this CLT together with a Berry-Esseen-type order of convergence $O(n^{-1/3})$ which is slightly worse than the order $O(n^{-1/2})$ in the classical CLT for sums of iid random variables on \mathbb{R} .

A CLT in the case $r > 1/2$ was treated earlier in in Section 3 of [V2] on the basis of a variant of Proposition 2.4. We here derive this CLT on the basis of Proposition 2.4. The reason for doing so is, that in our opinion, the proof of Proposition 2.4 above is more elementary than the corresponding variant in Section 3 of [V2], and that the present proof can be transfered without problems to a higher dimensional setting.

We finally turn to the case $r \in]0, 1/2[$ which was not handled before. It turns out that the cases $r \in]0, 1/6[$, $r = 1/6$, and $r \in]1/6, 1/2[$ lead to a different limit behavior.

Before going into details, we collect some properties of the moment functions m_k introduced in (2.4) from the literature.

4.1 Lemma. (1) m_1 is increasing on $[0, \infty[$.

(2) There is a constant $R > 0$ with $m_1(x) \leq Rx^2$ for all $x \geq 0$.

(3) For all $\mu, \nu \in M^1([0, \infty[)$ with finite first moment,

$$\int_0^\infty m_1 d(\mu * \nu) = \int_0^\infty m_1 d\mu + \int_0^\infty m_1 d\nu.$$

(4) For $k \in \mathbb{N}$ and $t \geq 0$, $m_k(x) \leq x^k$.

Proof. For (1), (3), and (4) see [Z2] or Section 7.2 of [BH]. Part (2) follows from the fact that m_1 is analytic with $m_1'(0) = 0$ in combination with Lemma 2.8. \square

4.2 Theorem. Let $\nu \in M^1([0, \infty[)$ with $\nu \neq \delta_0$ and finite second moment and $(S_n^{(\alpha, \beta, 1)})_n$ the associated Jacobi random walk without initial compression. For $k \in \mathbb{N}$ let

$$M_k := \int_0^\infty m_k d\nu(t) \in]0, \infty]$$

be the so called modified moments of ν with $M_1, M_2 < \infty$ by our assumption. Then

$$(S_n^{(\alpha, \beta, 1)} - nM_1)/\sqrt{n}$$

tends in distribution to $N(0, M_2 - M_1^2)$.

Moreover, if $\nu \in M^1([0, \infty[)$ in addition has third moments, then the distribution functions of $(S_n^{(\alpha, \beta, 1)} - nM_1)/\sqrt{n}$ tend uniformly to the distribution function of $N(0, M_2 - M_1^2)$ of order $n^{-1/3}$.

Proof. For the proof of the first statement on the basis of Corollary 2.9 we refer to Theorem 7.4.2 of [BH].

For the second statement fix a small constant $c > 0$ and put $T := cn^{1/3}$. Moreover, let $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ with $|\lambda| \leq T$. As ν has a finite third moment, the function $\tilde{\nu} : \mathbb{R} \rightarrow$

\mathbb{C} , $\lambda \mapsto \int_0^\infty \varphi_{i\rho-\lambda}^{(\alpha,\beta)}(t) d\nu(t)$ is three times differentiable (see [Z1] or 7.2.19 of [BH]), i.e., by the remainder in the Taylor expansion,

$$|\tilde{\nu}(\lambda/\sqrt{n}) - (1 + i\lambda M_1/\sqrt{n} - \lambda^2 M_2/(2n))| \leq \frac{\lambda^3 M_3}{6n^{3/2}}$$

for $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. Therefore, for suitable positive constants C_1, C_2, \dots ,

$$\left| e^{-i\lambda M_1/\sqrt{n}} \tilde{\nu}(\lambda/\sqrt{n}) - \left(1 - \frac{\lambda^2}{2n}(M_2 - M_1^2)\right) \right| \leq C_1 \frac{\lambda^3}{n^{3/2}},$$

and in particular, for $|\lambda| \leq T$ and $c > 0$ sufficiently small,

$$|\tilde{\nu}(\lambda/\sqrt{n})| \leq 1 - \frac{\lambda^2}{2n}(M_2 - M_1^2) + C_1 \frac{|\lambda|^3}{n^{3/2}} \leq 1 - \frac{C_2 \lambda^2}{n} \leq e^{-C_2 \lambda^2/n}.$$

Moreover, under this restriction and by the same arguments,

$$\left| e^{-\lambda^2(M_2 - M_1^2)/(2n)} - \left(1 - \frac{\lambda^2}{2n}(M_2 - M_1^2)\right) \right| \leq C_3 \frac{\lambda^4}{n^2}$$

and $|e^{-\lambda^2(M_2 - M_1^2)/(2n)}| \leq e^{-C_2 \lambda^2/n}$. Therefore, using $|a^n - b^n| \leq n|a - b| \cdot \max(|a|, |b|)^{n-1}$, we get

$$\begin{aligned} & \left| e^{-i\lambda M_1 \sqrt{n}} \tilde{\nu}(\lambda/\sqrt{n})^n - e^{-\lambda^2(M_2 - M_1^2)/2} \right| \\ & \leq n \cdot \left| e^{-i\lambda M_1/\sqrt{n}} \tilde{\nu}(\lambda/\sqrt{n}) - e^{-\lambda^2(M_2 - M_1^2)/(2n)} \right| \cdot C_4 \cdot e^{-C_5 \lambda^2} \\ & \leq C_6 \cdot \frac{\lambda^3}{n^{1/2}} \cdot e^{-C_5 \lambda^2}. \end{aligned} \tag{4.1}$$

On the other hand, by Corollary 2.9 and the multiplicativity of the Jacobi functions,

$$\tilde{\nu}(\lambda/\sqrt{n})^n = \int_0^\infty \varphi_{i\rho-\lambda}^{(\alpha,\beta)}(t) d\nu^{(n)}(t) = \int_0^\infty e^{i\lambda t} d\nu^{(n)}(t) + O(\lambda^2 + |\lambda|^3).$$

Therefore, by (4.1), the usual Fourier transform $\widehat{\nu^{(n)}}$ of $\nu^{(n)}$ satisfies

$$\begin{aligned} & \int_{-T}^T \frac{|\widehat{\nu^{(n)}}(\lambda/\sqrt{n}) \cdot e^{i\lambda M_1 \sqrt{n}} - e^{-\lambda^2(M_2 - M_1^2)/2}|}{|\lambda|} d\lambda \\ & \leq C_7 \int_{-T}^T \left(\frac{|\lambda|}{n} + \frac{|\lambda|^2}{n^{3/2}} + \frac{\lambda^2 + |\lambda|^3}{n^{1/2}} \cdot e^{-C_5 \lambda^2} \right) d\lambda \end{aligned} \tag{4.2}$$

As this expression is of order $O(n^{-1/3})$ for $T = cn^{1-1/3}$, we conclude from the lemma of Berry-Esseen (see, for instance, Lemma 2 in Section XVI.3 of Feller [Fe]) that the distribution functions of $(S_n^{(\alpha,\beta,1)} - nM_1)/\sqrt{n}$ tend uniformly to the distribution function of $N(0, M_2 - M_1^2)$ of order $n^{-1/3}$ as claimed. \square

4.3 Theorem. *Let $\nu \in M^1([0, \infty[)$ with $\nu \neq \delta_0$ and finite second moment $m_2 := \int_0^\infty x^2 d\mu(x) \in]0, \infty[$. Let $\alpha \geq \beta \geq -1/2$ with $\alpha > -1/2$, and let $r > 1/2$. Then*

$$\left(\frac{2(\alpha + 1)}{m_2} \right)^{1/2} n^{r-1/2} \cdot S_n^{(\alpha,\beta,n^{-r})}$$

tends in distribution to the Rayleigh distribution ρ_α with Lebesgue density

$$\frac{1}{2^\alpha \Gamma(\alpha + 1)} x^{2\alpha+1} e^{-x^2/2} \quad (x \geq 0).$$

The proof needs some preparations. The following estimation is proved in Lemma 3.5 of [V2] by using Lemma 4.1

4.4 Lemma. *There exists a constant $M = M(\alpha, \beta, r, \nu) > 0$ such that*

$$\mathbf{P}(S_n^{(\alpha, \beta, n^{-r})} \geq c) \leq \frac{M}{m_1(c) n^{2r-1}} \quad \text{for } c > 0, n \in \mathbb{N}.$$

For the rest of the proof of Theorem 4.3 we denote the distribution of $n^{r-1/2} S_n^{(\alpha, \beta, n^{-r})}$ by μ_n . The proofs of the following two lemmas are similar to, but slightly different from those of Lemmas 3.6 and 3.7 of [V2].

4.5 Lemma. *For all $\lambda \geq 0$,*

$$\lim_{n \rightarrow \infty} \int_0^\infty \left| j_\alpha(\lambda t) - \varphi_{\lambda n^{r-1/2}}^{(\alpha, \beta)}(t n^{1/2-r}) \right| d\mu_n(t) = 0.$$

Proof. Let $c > 0$. Then, by Proposition 2.4, the boundedness of the involved Jacobi and Bessel functions and by the preceding lemma,

$$\begin{aligned} A_n &:= \int_0^\infty \left| j_\alpha(\lambda t) - \varphi_{\lambda n^{r-1/2}}^{(\alpha, \beta)}(t n^{1/2-r}) \right| d\mu_n(t) = \left(\int_0^c + \int_c^\infty \right) \left| \dots \right| d\mu_n(t) \\ &\leq \frac{H(c)|\lambda|}{n^{r-1/2}} + 2 \cdot \mathbf{P}(n^{r-1/2} S_n^{(\alpha, \beta, n^{-r})} \geq c) \\ &\leq \frac{H(c)|\lambda|}{n^{r-1/2}} + \frac{2M}{m_1(c n^{1/2-r}) \cdot n^{2r-1}} \end{aligned} \quad (4.3)$$

for some constants M and some $H(c)$ depending on c . On the other hand, as $m_1''(0) > 0$, there exist $a, b > 0$ with $m_1(x) \geq ax^2$ for $x \in [0, b]$.

Now let $\varepsilon > 0$. Then choose c with $2M/(ac^2) \leq \varepsilon/2$, and now n large enough with

$$\frac{H(c)|\lambda|}{n^{r-1/2}} \leq \varepsilon/2 \quad \text{and} \quad c n^{1/2-r} \leq b.$$

As the latter implies

$$\frac{2M}{m_1(c n^{1/2-r}) \cdot n^{2r-1}} \leq \frac{2M}{ac^2} \leq \varepsilon/2,$$

we obtain from (4.3) that $A_n \leq \varepsilon$ for large n as claimed. \square

4.6 Lemma. *Let $\lambda > 0$ and $\nu \in M^1([0, \infty[)$ with finite second moment $m_2 < \infty$. Then, for $n \rightarrow \infty$,*

$$\int_0^\infty \varphi_{\lambda n^{r-1/2}}^{(\alpha, \beta)}(t/n^r) d\nu(t) = 1 - \frac{\lambda^2 m_2}{4(\alpha + 1)n} + o(1/n).$$

Proof. Consider $H_\lambda(t) := |\varphi_\lambda^{(\alpha,\beta)}(t) - j_\alpha(\lambda t)| \leq 2$. We apply Proposition 2.4 to $t/n^r \leq 1$ instead of t with $n = 1$ there. Therefore, for some $C > 1$, and by Markov's inequality,

$$\begin{aligned} \int_0^\infty H_{\lambda n^{r-1/2}}(t/n^r) d\nu(t) &\leq \int_0^{n^r} H_{\lambda n^{r-1/2}}(t/k^r) d\nu(t) + 2\nu([n^r, \infty[) \\ &\leq C\lambda \cdot \int_0^{n^r} \frac{t^2 n^{r-1/2}}{n^{2r}} d\nu(t) + 2\nu([n^r, \infty[) \\ &\leq C\lambda m_2/n^{r+1/2} + 2m_2/n^{2r} = o(1/n). \end{aligned} \quad (4.4)$$

Furthermore, as $m_2 < \infty$, the Hankel transform $g(\lambda) := \int_0^\infty j_\alpha(\lambda t) d\nu(t)$ of ν is two-times differentiable at $\lambda = 0$ with

$$g(\lambda) = 1 - \frac{\lambda^2 m_2}{4(\alpha + 1)} + o(\lambda^2) \quad \text{for } \lambda \rightarrow 0$$

(see Theorem 4.7 of [Z1] or Section 7.2 of [BH]) with $m_1 = 0$ and $m_2(x) = x^2/(\alpha + 1)$ there). Hence, by (4.4),

$$\begin{aligned} \int_0^\infty \varphi_{\lambda n^{r-1/2}}^{(\alpha,\beta)}(t/n^r) d\mu(t) &= \int_0^\infty j_\alpha(\lambda t/n^{1/2}) d\mu(t) + o(1/n) \\ &= 1 - \frac{x^2 \lambda^2}{4(\alpha + 1)n} + o(1/n) \end{aligned} \quad (4.5)$$

as claimed. \square

Proof of Theorem 4.3. Fix $\lambda \in [0, \infty[$. Then Lemmas 4.5 and 4.6 lead to

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty j_\alpha(\lambda t) d\mu_n(t) &= \lim_{n \rightarrow \infty} \int_0^\infty \varphi_{\lambda n^{r-1/2}}^{(\alpha,\beta)}(t n^{1/2-r}) d\mu_n(t) \\ &= \lim_{n \rightarrow \infty} \int_0^\infty \varphi_{\lambda n^{r-1/2}}^{(\alpha,\beta)}(t) d\nu_{n^{-r}}^{(n)}(t) = \lim_{n \rightarrow \infty} \left(\int_0^\infty \varphi_{\lambda n^{r-1/2}}^{(\alpha,\beta)}(t) d\nu_{n^{-r}}(t) \right)^n \\ &= \lim_{n \rightarrow \infty} \left(\int_0^\infty \varphi_{\lambda n^{r-1/2}}^{(\alpha,\beta)}(t/n^r) d\nu(t) \right)^n = \lim_{n \rightarrow \infty} \left(1 - \frac{x^2 m_2}{4(\alpha + 1)n} + o(1/n) \right)^n \\ &= e^{-\lambda^2 m_2/4(\alpha+1)}. \end{aligned} \quad (4.6)$$

As the Rayleigh distribution ρ_α satisfies

$$\int_0^\infty j_\alpha(\lambda t) d\rho_\alpha(t) = e^{-\lambda^2/2},$$

Levy's continuity theorem for the Hankel transform (see, for instance, Section 4.2 of [BH]) now yields that $\left(\frac{2(\alpha+1)}{m_2}\right)^{1/2} n^{r-1/2} \cdot S_n^{(\alpha,\beta,n^{-r})}$ tends to ρ_α as claimed. \square

4.7 Theorem. Let $\nu \in M^1([0, \infty[)$ with $\nu \neq \delta_0$, with compact support, and thus with finite moments $m_k := \int_0^\infty x^k d\mu(x) \in]0, \infty[$ ($k \in \mathbb{N}$). Let $\alpha \geq \beta \geq -1/2$ with $\alpha > -1/2$ and $r \in]0, 1/2[$. Then the random variables $S_n^{(\alpha,\beta,n^{-r})}$ have the following behavior for $n \rightarrow \infty$.

(1) If $r \in]1/6, 1/2[$, then

$$\frac{S_n^{(\alpha, \beta, n^{-r})} - \frac{\rho m_2}{2(\alpha+1)} \cdot n^{1-2r}}{n^{1/2-r}}$$

tends in distribution to $N(0, \frac{m_2}{2(\alpha+1)})$.

(2) If $r = 1/6$, then

$$\frac{S_n^{(\alpha, \beta, n^{-1/6})} - \frac{\rho m_2}{2(\alpha+1)} \cdot n^{2/3}}{n^{1/3}}$$

tends in distribution to $N(-\frac{\rho(\alpha+3\beta+2)m_4}{12(\alpha+1)(\alpha+2)}, \frac{m_2}{2(\alpha+1)})$.

(3) If $r \in]0, 1/6[$, then

$$\frac{S_n^{(\alpha, \beta, n^{-r})} - \frac{\rho m_2}{2(\alpha+1)} \cdot n^{1-2r}}{n^{1-4r}} \longrightarrow -\frac{\rho(\alpha+3\beta+2)m_4}{12(\alpha+1)(\alpha+2)}$$

in probability.

Notice that the drift term $-\frac{\rho(\alpha+3\beta+2)m_4}{12(\alpha+1)(\alpha+2)}$ in (2) and (3) is negative, and that the case (2) combines the features of the cases (1) and (3).

The proof will be based on a simple Taylor-type expansion of of the $\varphi_\lambda^{(\alpha, \beta)}$ which is an immediate consequence of the well-known representation of $\varphi_\lambda^{(\alpha, \beta)}$ as hypergeometric series.

4.8 Lemma. Let $\alpha \geq \beta \geq -1/2$ with $\alpha > -1/2$, and let $\lambda \in \mathbb{R}$ and $t \geq 0$. Then, for $a, r > 0$ and $n \rightarrow \infty$,

$$\begin{aligned} \varphi_{i\rho-\lambda/n^a}^{(\alpha, \beta)}(t/n^r) &= 1 + \frac{i\rho\lambda t^2}{2(\alpha+1)n^{a+2r}} - \frac{\lambda^2 t^2}{4(\alpha+1)n^{2a+2r}} \\ &\quad - \frac{i\rho(\alpha+3\beta+2)t^4\lambda}{12(\alpha+1)(\alpha+2)n^{a+4r}} + O(n^{-a-6r}) + O(n^{-2a-4r}) \end{aligned}$$

locally uniformly in $t \in [0, \infty[$.

Proof. Using $\rho = \alpha + \beta + 1$ and Eq. (2.4) of [Ko3], we have

$$\begin{aligned} \varphi_{i\rho-\lambda/n^a}^{(\alpha, \beta)}(t/n^r) &= {}_2F_1\left(\rho + i\lambda/(2n^a), -i\lambda/(2n^a); \alpha + 1; -\text{sh}^2(t/n^r)\right) \\ &= 1 + \frac{(\rho + i\lambda/(2n^a)) \cdot i\lambda}{2n^a \cdot (\alpha + 1)} \cdot \text{sh}^2(t/n^r) \\ &\quad - \frac{(\rho + i\lambda/(2n^a))(\rho + 1 + i\lambda/(2n^a))i\lambda(1 - i\lambda/(2n^a))}{4n^a \cdot (\alpha + 1)(\alpha + 2)} \cdot \text{sh}^4(t/n^r) \\ &\quad + O(n^{-a-6r}) \\ &= 1 + \frac{(\rho + i\lambda/(2n^a)) \cdot i\lambda}{2n^a \cdot (\alpha + 1)} \cdot \left(\frac{t}{n^r} + \frac{t^3}{6n^{3r}}\right)^2 \\ &\quad - \frac{(\rho + i\lambda/(2n^a))(\rho + 1 + i\lambda/(2n^a))i\lambda(1 - i\lambda/(2n^a))}{4n^a \cdot (\alpha + 1)(\alpha + 2)} \cdot \frac{t^4}{n^{4r}} \\ &\quad + O(n^{-a-6r}) + O(n^{-2a-4r}) \end{aligned}$$

locally uniformly in t which readily leads to the claim. \square

This expansion leads immediately to:

4.9 Corollary. *Let $\nu \in M^1([0, \infty[)$ with compact support. Then, in the setting of the preceding lemma,*

$$\begin{aligned} \int_0^\infty \varphi_{i\rho-\lambda/n^a}^{(\alpha,\beta)}(t/n^r) d\nu(t) = & 1 + \frac{i\rho\lambda m_2}{2(\alpha+1)n^{a+2r}} - \frac{\lambda^2 m_2}{4(\alpha+1)n^{2a+2r}} \\ & - \frac{i\rho(\alpha+3\beta+2)m_4\lambda}{12(\alpha+1)(\alpha+2)n^{a+4r}} + O(n^{-a-6r}) + O(n^{-2a-4r}). \end{aligned}$$

Proof of Theorem 4.7. We first recapitulate for all cases that by Corollary 2.9,

$$|\varphi_{i\rho-\lambda}^{(\alpha,\beta)}(t) - e^{i\lambda \cdot t}| = O(\lambda^2 + |\lambda|^3) \quad (4.7)$$

uniformly in t . We now consider the different cases:

- (1) Let $r \in]1/6, 1/2[$. In this case we put $a := 1/2 - r \in]0, 1/3[$ and observe that, due to $r \geq 1/6$, $1 = 2a + 2r < a + 4r$. Therefore, Eq. (4.7), the definition of the measures $\nu_{n^{-r}}$ above, the multiplicativity of Jacobi functions, and the preceding corollary imply that for all $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$, the classical Fourier transform of the distribution $\mu_n := \nu_{n^{-r}}^{(n)}$ of the random variable $S_n^{(\alpha,\beta,n^{-r})}$ satisfies

$$\begin{aligned} \hat{\mu}_n(\lambda/n^a) &= \int_0^\infty e^{-it\lambda/n^a} d\nu_{n^{-r}}^{(n)}(t) \\ &= \int_0^\infty \varphi_{i\rho+\lambda/n^a}^{(\alpha,\beta)}(t) d\nu_{n^{-r}}^{(n)}(t) + o(1) \\ &= \left(\int_0^\infty \varphi_{i\rho+\lambda/n^a}^{(\alpha,\beta)}(t) d\nu_{n^{-r}}(t) \right)^n + o(1) \\ &= \left(\int_0^\infty \varphi_{i\rho+\lambda/n^a}^{(\alpha,\beta)}(t/n^r) d\nu(t) \right)^n + o(1) \\ &= \left(1 - \frac{i\rho\lambda m_2}{2(\alpha+1)n^{a+2r}} - \frac{\lambda^2 m_2}{4(\alpha+1)n} + o(1/n) \right)^n + o(1). \end{aligned}$$

Therefore, using $a = 1/2 - r$,

$$\begin{aligned} \hat{\mu}_n(\lambda/n^a) \cdot \exp\left(\frac{i\rho m_2 \lambda n^{1/2-r}}{2(\alpha+1)}\right) &= \left(1 - \frac{i\rho\lambda m_2}{2(\alpha+1)n^{1/2+r}} - \frac{\lambda^2 m_2}{4(\alpha+1)n} + o(1/n) \right)^n \\ &\quad \cdot \left(1 + \frac{i\rho\lambda m_2}{2(\alpha+1)n^{1/2+r}} + O(n^{-1-2r}) \right)^n + o(1) \\ &= \left(1 - \frac{\lambda^2 m_2}{4(\alpha+1)n} + o(1/n) \right)^n + o(1) \end{aligned}$$

which tends for $\lambda \in \mathbb{R}$ to $\exp(-\lambda^2 m_2/(4(\alpha+1)))$. Levy's continuity theorem for the classical Fourier transform now implies that

$$\frac{S_n^{(\alpha,\beta,n^{-r})} - \frac{\rho m_2}{2(\alpha+1)} \cdot n^{1-2r}}{n^{1/2-r}}$$

tends in distribution to $N(0, \frac{m_2}{2(\alpha+1)})$ as claimed.

- (2) Now let $r = 1/6$. Here we put $a := 1/6$. Then $2a + 2r = a + 4r = 1$ and $a + 2r = 2/3$. As in the first case, we obtain from the preceding corollary

$$\begin{aligned} & \hat{\mu}_n(\lambda/n^{1/3}) \cdot \exp\left(\frac{i\rho m_2 \lambda n^{1/3}}{2(\alpha+1)}\right) \\ &= \left(1 - \frac{i\rho \lambda m_2}{2(\alpha+1)n^{2/3}} - \frac{\lambda^2 m_2}{4(\alpha+1)n} + \frac{i\rho(\alpha+3\beta+2)\lambda m_4}{12(\alpha+1)(\alpha+2)n} + o(1/n)\right)^n \\ & \quad \cdot \left(1 + \frac{i\rho \lambda m_2}{2(\alpha+1)n^{2/3}} + O(n^{-4/3})\right)^n + o(1) \end{aligned}$$

which tends for $\lambda \in \mathbb{R}$ to $\exp\left(-\frac{\lambda^2 m_2}{4(\alpha+1)} + \frac{i\rho(\alpha+3\beta+2)\lambda m_4}{12(\alpha+1)(\alpha+2)}\right)$. The proof is now completed as before.

- (3) Let $r \in]0, 1/6[$. We here put $a := 1 - 4r \in]1/3, 1[$, and obtain $2a + 2r > a + 4r = 1$ and $a + 2r = 1 - 2r$. As in the first and second part, we obtain

$$\begin{aligned} & \hat{\mu}_n(\lambda/n^a) \cdot \exp\left(\frac{i\rho m_2 \lambda n^{2r}}{2(\alpha+1)}\right) \\ &= \left(1 - \frac{i\rho \lambda m_2}{2(\alpha+1)n^{1-2r}} + \frac{i\rho(\alpha+3\beta+2)\lambda m_4}{12(\alpha+1)(\alpha+2)n} + o(1/n)\right)^n \\ & \quad \cdot \left(1 + \frac{i\rho \lambda m_2}{2(\alpha+1)n^{1-2r}} + O(n^{-2+4r})\right)^n + o(1) \end{aligned}$$

which tends for $\lambda \in \mathbb{R}$ to $\exp\left(\frac{i\rho(\alpha+3\beta+2)\lambda m_4}{12(\alpha+1)(\alpha+2)}\right)$. Therefore, again by Levy's continuity theorem,

$$\frac{S_n^{(\alpha, \beta, n^{-r})} - \frac{\rho m_2}{2(\alpha+1)} \cdot n^{1-2r}}{n^{1-4r}} \longrightarrow -\frac{\rho(\alpha+3\beta+2)m_4}{12(\alpha+1)(\alpha+2)}$$

in distribution and thus in probability.

□

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